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3 SEM TDC MTMH (CBCS) C 5

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(Held in April–May, 2021)

MATHEMATICS

(Core)

Paper : C–5

(Theory of Real Functions)

Full Marks : 80

Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. (a) Define cluster point of a set A. 1

(b) Using the definition of limit to show that

$$\lim_{x \quad 2} (x^2 \quad 4x) \quad 12$$ 2

(c) If a set $A \quad R$ and $f : A \quad R$ has a limit at $C \quad R$, then prove that f is bounded on some neighbourhood of C. 3

(d) Evaluate the following limit (any *one*) : 3

(i) $\lim_{x} \frac{\sqrt{x} \quad 5}{\sqrt{x} \quad 3}, \quad x \quad 0$

(ii) $\lim_{x \quad 0} \frac{\sqrt{1 \quad 2x} \quad \sqrt{1 \quad 3x}}{x \quad 2x^2}, \quad x \quad 0$

(e) Let I be an interval and let $f : I \quad R$ be continuous on I, if $a, b \quad I$ and if $k \quad R$ satisfies $f(a) \quad k \quad f(b)$, then there exists a point $c \quad I$ between a and b, prove that $f(c) \quad k$. 4

Or

State and prove preservation of intervals theorem.

2. (a) Prove that the constant function $f(x) \quad b$ is continuous on the set of real number R. 1

(b) Write the type of discontinuity if

$$\lim_{x \quad c} f(x)$$

exists but not equal to $f(c)$. 1

(c) Define uniform continuity of a function. 2

(d) Let $A \quad R$, let f and g be two continuous functions at $x \quad c$ on A to R. Prove that $f \quad g$ is continuous at $x \quad c$. 3

3. *(a)* A function $f : A \quad R$ is continuous at the point $c \quad A$ and if for every sequence $\{x_n\}$ in A that converges to c. Prove that the sequence $\{f(x_n)\}$ converges to $f(c)$. 5

Or

State and prove location of roots theorem.

(b) Let I be a closed bounded interval and let $f : I \quad R$ be continuous on I. Then prove that f is uniformly continuous on I. 5

Or

Test the following function for continuity at $x \quad 0$:

$$f(x) \quad \begin{matrix} 0 & , & x \quad 0 \\ x \sin \frac{1}{x} & , & x \quad 0 \end{matrix}$$

4. *(a)* If a function f is differentiable at c, then choose the correct answer : 1

(i) $\dfrac{f(b) \quad f(a)}{b \quad a} \quad f(c)$

(ii) $f(c) \quad \lim\limits_{x \quad c} \dfrac{f(x) \quad f(c)}{x \quad c}$, provided limit exists

(iii) $\lim\limits_{x \quad c} f(x) \quad f(c)$

(iv) $\lim\limits_{x \quad c} f(x) \quad \lim\limits_{x \quad c} f(x)$

(b) If $f : I \quad R$ has a derivative at $c \quad I$, then prove that f is continuous at c. 2

(c) Let c be an interior point of the interval I at which $f : I \quad R$ has relative maximum. If the derivative of f at c exists, then prove that $f(c) \quad 0$. 3

(d) State and prove Caratheodory's theorem. 4

Or

Prove that if $f : R \quad R$ is an even function and has a derivative at every point, then f is an odd function and vice versa.

5. *(a)* Find the derivative of

$$f(x) \quad \sqrt{5 \quad 2x \quad x^2}$$ 1

(b) Write the statement of Rolle's theorem. 2

(c) If $f(x)$ and $g(x)$ are continuous on $I \quad [a, b]$, they are differentiable on (a, b) and $f(x) \quad g(x)$ for all $x \quad (a, b)$, then there exists a constant k, prove that $f(x) \quad g(x) \quad k$ on I. 3

(d) State and prove Lagrange's mean value theorem. 5

Or

If f is differentiable on $I \quad [a, b]$ and if k is a number between $f(a)$ and $f(b)$, then there is at least one point c in (a, b). Prove that $f(c) \quad k$.

(e) Applying mean value theorem, prove that $x \quad \sin x \quad x$, for $x \quad 0$. 4

Or

Verify Rolle's theorem for the following function :

$$f(x) \quad x^3 \quad 6x^2 \quad 11x \quad 6, \; x \quad [1, 3]$$

6. *(a)* Write the remainder after n terms of Taylor's theorem in Cauchy's form. 1

(b) Deduce mean value theorem from Cauchy's mean value theorem. 2

(c) Verify Cauchy's mean value theorem for the functions $f(x) \quad x^2$, $g(x) \quad x^3$ in the interval $[1, 2]$. 4

(d) Let $I \quad R$ be an open interval, let $f : I \quad R$ be differentiable on I and $f \quad (a)$ exists at $a \quad I$. Show that

$$\lim_{h \quad 0} \frac{f(a \quad h) \quad 2f(a) \quad f(a \quad h)}{h^2} \quad f \quad (a)$$ 5

Or

State and prove Cauchy's mean value theorem.

7. *(a)* Write the Maclaurin's series for the expansion of $f(x)$ as a power series in x. 1

(b) Define convex function. 2

(c) Expand the following by using Maclaurin's theorem in an infinite series in powers of x (any *one*) : 4

(i) e^x

(ii) $\sin x$

(d) State and prove Taylor's theorem with Lagrange's form of remainder. 6

Or

Let I be an open interval and let $f : I \quad R$ have a second derivative on I. Then prove that f is a convex function on I if and only if $f \ (x) \quad 0$, for all $x \quad I$.

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