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1 SEM TDC MTMH (CBCS) C 2

2021

(March)

MATHEMATICS

(Core)

Paper : C–2

(Algebra)

Full Marks : 80
Pass Marks : 32

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. (a) State the argument of the complex number $\sin\theta + i\cos\theta$. 1

(b) Reduce the complex number $1+\cos\alpha + i\sin\alpha$ into polar form. 2

(c) Find all the values of $(1+i)^{\frac{1}{5}}$. 3

(d) If

$$\sin\alpha + \sin\beta + \sin\gamma = 0 = \cos\alpha + \cos\beta + \cos\gamma$$

then show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$$
$$= \sin^2\alpha + \sin^2\beta + \sin^2\gamma$$ 4

Or

State and prove De Moivre's theorem for positive integers.

2. (a) State the principle of mathematical induction. 1

(b) For two functions f and g, both their composites $g \circ f$ and $f \circ g$ exist such that $g \circ f = I = f \circ g$, where I is the identity function. Consider the following statements :

(1) f is one-one and onto

(2) g is onto but not one-one

Choose the correct answer. 1

(i) (1) is true and (2) is false

(ii) (1) is false and (2) is true

(iii) Both the statements (1) and (2) are true

(iv) Both the statements (1) and (2) are false *(c)* Investigate whether the map $f:\mathbb{Z}\to\mathbb{Z}$ defined by $f(x)=3x$ is a bijection or not, where $\mathbb{Z}$ denotes the set of integers. 2

(d) Find the remainder when 2^{50} is divided by 7. 2

(e) Let A be the set of all lines in a plane. Define a relation R in A as $R=\{(l, m): l, m\in A, l||m\}$. Show that R is an equivalence relation ($l||m$ means l is parallel to m). 3

(f) If $a\equiv b \pmod{n}$, show that

$$\gcd(a, n)=\gcd(b, n)$$ 3

(g) Given $f:X\to Y$ and $g:Y\to Z$ are bijections. Show that $g\circ f:X\to Z$ is also a bijection. 4

Or

Show that $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=3x+2$ is a bijection and find its inverse.

(h) For $a, b\in\mathbb{N}$, show that

$$\gcd(a, b)\times \operatorname{lcm}(a, b)=ab$$ 4

(i) If a and b are integers and $b\neq 0$, show that there exists unique integers q and r such that $a=bq+r$, where $-\frac{|b|}{2}<r\leq\frac{|b|}{2}$. 5

3. *(a)* Consider the following two statements :

(1) Two fundamental questions about a linear system involve existence and uniqueness.

(2) Two matrices of the same type are row equivalent if they have the same number of rows.

State which of the following is true. 1

(i) (1) $\Rightarrow$ (2)

(ii) (2) $\Rightarrow$ (1)

(iii) (1) is true and (2) is false

(iv) (1) is false and (2) is true

(b) Identify the pivot columns of the following matrix : 1

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

(c) Show that

$$u+(v+w)=(u+v)+w \quad \forall\, u, v, w\in\mathbb{R}^n$$ 2

(d) For the vectors

$$a=\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, b=\begin{bmatrix} 5 \\ -13 \\ 3 \end{bmatrix} \text{ and } c=\begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

determine whether $c\in\operatorname{span}\{a, b\}$ or not. 2

(e) Prove that a set $\{v_1, \cdots, v_p\}$ in $\mathbb{R}^n$ is linearly dependent, if $p > n$. 2

(f) Investigate, for what value of h, the vector

$$\begin{bmatrix} -2 \\ 3 \\ h \end{bmatrix}$$

is a linear combination of

$$\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} \text{ and } \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$ 3

(g) The solution set of a linear non-homogeneous system $Ax = b$ is given by $x_1 = 3x_4$, $x_2 = 8 + x_4$, $x_3 = 2 - 5x_4$ with x_4 free. State the solution set in the form $w = p + v_p$, where p is a solution of the system $Ax = b$ and v_h is a vector in the solution set of $Ax = 0$. Give the geometrical interpretation of the solution of $Ax = b$. 2+2=4

Or

If the system $Ax = b$ has a solution, explain why the solution is unique when $Ax = 0$ has only trivial solution. 4

(h) Give the geometrical interpretations of span $\{v\}$ and span $\{u, v\}$, where $u, v \in \mathbb{R}^n$. Determine whether the vectors

$$\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

are linearly independent or not. 1+1+3=5

4. *(a)* State whether True or False : 1

Corresponding to every linear transformation there exists a matrix transformation.

(b) Show that a transformation T is linear if and only if

$$T(cu + dv) = c\,T(u) + d\,T(v) \quad \forall\, u, v$$

in the domain of T and $\forall$ scalars c and d. 3

(c) Define an invertible matrix. Show that if A is an invertible $n \times n$ matrix, then $\forall\, b \in \mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$. 1+2=3

(d) Define null space of a matrix A. Show that null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$. 1+2=3

(e) Row reduce the following matrix into echelon form :

$$\begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$$

Indicate the pivot columns and hence, state its rank. 2+1=3

(f) Row reduce the augmented matrix $[A \;\; I]$, where

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & 1 \\ 1 & -1 & -2 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Show that $[A \;\; I]$ is row equivalent to $[I \;\; A^{-1}]$ and state the value of A^{-1}. 4

(g) Given

$$T(x_1, x_2) = (2x_1 + x_2, 3x_1 + 5x_2, x_1 + 7x_2)$$

find the standard matrix represented by T. Show that T is one-one but not onto. 2+2=4

(h) Find a basis for the eigenspace corresponding to the eigenvalue $\lambda = 1$ of the matrix

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

4

Or

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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